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Torsional-Flexural Buckling of Unevenly Battened Columns under Eccentrical Compressive Loading

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Abstract: In this paper, an analytical model is developed to determine the torsional-flexural buckling load of a channel column braced by unevenly distributed batten plates. Solutions of the critical buckling loads were derived for three boundary cases using the energy method in which the rotating angle between the adjacent battens was presented in the form of a piecewise cubic Hermite interpolation (PCHI) for unequally spaced battens. The validity of the PCHI method was numerically verified by the classic analytical approach for evenly battened columns and a finite element analysis for unevenly battened ones, respectively. Parameter studies were then performed to examine the effects of loading eccentricities on the torsional-flexural buckling capacity of both evenly and unevenly battened columns. Design parameters taken into account were the ratios of pure torsional buckling load to pure flexural buckling load, the number and position of battens, and the ratio of the relative extent of the eccentricity. Numerical results were summarized into a series of relative curves indicating the combination of the buckling load and corresponding moments for various buckling ratios.

Keyword: Torsional-flexural buckling, Critical buckling load, PCHI, Battened column, Unevenly

Introduction

Battened built-up columns are widely used in steel buildings and bridges. They help reduce the effective length of the whole structure, and enhance the corresponding local and global buckling capacities. Generally, these battened columns are made of one or two longitudinal members with plates (batten plates) connecting two parallel components such as flanges at specific positions along the length of the column, forming several units of completely closed cross-sections, which locally stiffen the members with enhanced shear and bending resistance.

In design of battened columns, two features are taken into account, which differentiate them from other structural members. The first one is the shear deformation effects of battened columns on the global buckling capacity. The calculation of the compressive strength of built-up columns, based on the approach proposed by Engesser (1891), was presented by Bleich (1952), Timoshenko and Gere (1961), Bazant and Cedolin (1991). The stiffeners (batten plates) were replaced by a piece of equivalent plate for simplicity. The lateral displacement of this equivalent plate is affected by not only the bending stress in the column, but also the transverse shear stress. The latter is taken into account by the introduction of the equivalent slenderness ratio from which the global buckling capacity can be obtained. Tests were conducted on battened columns under eccentrically applied axial compression using specimens with various sized batten plates by Hashemia and Jafarib (2009). The simultaneous effects of the bending and shear deformations and semi-rigid connections (Xu and Grierson 1993) were considered for the elastic critical buckling load by Aristizabal-Ochoa (2004, 2011). These studies showed that these effects must be taken into account particularly in structures made of columns with relatively low effective shear areas A_s such as battened and laced columns. More recently Chen and Li (2013) considered the shear effects of evenly distributed battens on the global buckling capacity.

The second issue is the discrete effect of battens on the global buckling capacity of the columns. Most of the studies have been on equally braced columns. The torsional-flexural buckling capacity of a channel column braced by evenly distributed batten plates was analyzed theoretically by Wang (1985) using the Rayleigh-Ritz method and validated experimentally. Liu et al. (2009) provided an experimental verification of the American Institute of Steel Construction (AISC)slenderness ratio formulas (2005) for built-up compressive members. Based on generalized beam theory (GBT), Basaglia and Camotim (2015) assessed how different geometries and bracing arrangements affect the local, distortional and global buckling behavior of four structural systems. El Aghoury et al (2010) tested variation of the width–thickness ratio as well as the short and medium slenderness of evenly battened columns and showed that the interaction between the width–thickness ratios and the overall column slenderness decreases the strength of battened columns. Experiments on buckling of intermediately braced C-channels were conducted and compared with the theoretical buckling capacities by Whittle and Ramseyer (2009). The validity of Ayrton–Perry (1886) and ultimate strength methods were tested considering the equal spacing and dimensions of batten plates, and the initial imperfections in batten columns by Hashemia and Jafarib (2012). Evenly battened columns were also taken as statically indeterminate structures where the global buckling problem was reduced to a two-point boundary value problem numerically, the critical force of which can be determined as the smallest eigenvalue of a fourth-order system of homogeneous linear algebraic equations (Razdolsky 2014).

Yet, the discrete feature of batten plates in unevenly battened columns is rarely seen in the literature. Using an extended Dirac function Takabatake (1988) express the flexural and torsional stiffness of battened I-beams, in which the rotation function remains the same. The effective length coefficients were then introduced by Takabatake et al. (1991) with the comparison between the theoretical results and experimental ones. However, these studies focused on only the increased flexural and torsional stiffness produced by battens, and not the changed rotation function during buckling. Buckling caused by axially compression was also not analyzed.

Considering the potential to enhance the global buckling capacity of columns by optimizing the spacing between batten plates without increasing their numbers, the main objective of this work is to develop an analytical procedure using the energy method to determine the torsional-flexural buckling critical load of unevenly battened columns. For considering the discrete effect of battens, the rotating angle between adjacent battens is described using the piecewise cubic Hermite interpolation (PCHI). Solutions of the critical buckling load were developed for different boundary conditions. The appropriation and efficiency of the PCHI solutions were numerically verified by comparing with the classic analytical approach (Wang 1985) for the case of evenly battened columns and with the finite element analysis using conventional shell element model for the case of unevenly battened columns, respectively. In order to consider the effects of loading eccentricities on the torsional-flexural buckling load, parametric studies were performed. The design parameters were the ratios of the pure torsional buckling load to pure flexural buckling load, the number and position of battens, and the ratio of the relative extent of the eccentricity. As a result, the relationship of the critical buckling load P_{cr} and the corresponding moment M_{cr} produced by P_{cr} for various buckling ratios were depicted and discussed. The present analytical solution highlights the importance of taking into account of the unevenly distributed battens on the mixed torsional-flexural buckling of battened columns.

Structural model

Consider a channel column with a single symmetric axis. The column is braced by distributed batten plates attached to the free edges of channel flanges as illustrated in Fig. 1. The centroidal axis *OO'* is set as the longitudinal z-axis. And the x- and y-axis correspond to the principal bending axes with the coordinate system origin O set at one end of the column. Functions $u(z)$ and $v(z)$ are displacements along x- and y-axis, measured from the shear center axis *SS'* located at (x_0, y_0). The rotating angle about the shear center axis *SS'* is denoted as $\varphi(z)$.

For analysis, the column is assumed to be (i) made of a homogeneous, isotropic and linearly elastic material with the Young's and shear moduli E and G, respectively; (ii) constrained by different boundary conditions presented by u, v, φ and their first- and second-order derivatives as described in Table 1; (iii) loaded by an axially compressive load P with eccentricities e_x and e_y. The overall length of the column is L. The thickness and width of the flanges are t_1 and b, while the height of the web, measured between the mid-lines of the top and bottom flanges, is h and the thickness is t_2. The batten plates, all of equal size with thickness t_p and width b_p, are fixed to the open side of channel flanges to form several closed sections intermittently along the z-axis. The positions of the mid-line of these batten plates are marked by z_j measured from O. The mid-line distance between the neighboring battens is:

$$h_j = z_{j+1} - z_j \quad (1 \leq j \leq n-1) \tag{1}$$

n being the total number of batten plates. Note that h_j is not a constant for an unevenly battened column.

Three different boundary conditions (BC) presented by u, v, φ and their first- and second-order derivatives are considered corresponding to hinged supports with and without warping constraints at both ends, and fixed supports at both ends, as listed in Table 1. Here functions φ for all BCs in Table 1 are the rotation functions of unbattened columns.

For analyzing the global buckling mode of the column, the following assumptions are made:

(1) Based on the Vlasov assumption (Vlasov 1961), the shear deformation and residual stresses of the column are not considered and the cross-section of the column remains a plane experiencing no distortion throughout the buckling process (Andrade et al. 2010). This implies that the effect of distortion buckling and the localized flange deformation (Harrison 1974) is not allowed.

(2) The effect of the pre-buckling deflection is negligible, and the centroidal axis *OO'* and the shear center axis *SS'* remain in line until global buckling occurs.

(3) The function φ at the mid-line of each batten plate along the z-axis remains the same.

(4) Following (Szewczak et al. 1983) that the torsional stiffness of an open-sectional member with intermediate stiffeners is not sensitive to the stiffener thickness, the dimensions of a batten plate, ie. its thickness t_p and width b_p, are not considered in the global buckling mode of a battened column, as long as the shearing stiffness produced by batten plates remains large enough to resist warping deformations. Based on this assumption, the rotation function φ satisfies the following additional condition on the mid-line of each batten plate:

$$\varphi'(z) = 0 \text{ at } z = z_j \, (1 \leq j \leq n) \tag{2}$$

(5) The effects of the number, spacing and dimensions of batten plates are considered insignificant on the pure flexural deformations of a battened column, which implies that functions $u(z)$ and $v(z)$ remain the same for channel columns with and without batten plates, respectively. In fact,

this means no excessive reinforcement of the column by a large number of batten plates for the flexural rigidity of battened columns.

Based on the above assumptions, the solution for the global buckling of unevenly battened columns rests in finding appropriate φ to satisfy the prescribed boundary conditions.

Discussion for the rotation function φ

The global buckling of a channel column braced by n evenly battens was studied under different boundary conditions in (Wang 1985). For BC Case I, the analytical solution of the rotation function φ has the form:

$$\varphi = 1 - \cos\frac{2(i-1)a\pi}{L} + \sin\frac{a\pi}{L}\sin\frac{(2i-1)a\pi}{L}\left[1+(-1)^i\cos\frac{\pi z}{a}\right] \tag{3a}$$

for n being an odd number. And for n being an even number

$$\varphi = \begin{cases} 1 - \cos\frac{2(i-1)a\pi}{L} + \sin\frac{a\pi}{L}\sin\frac{(2i-1)a\pi}{L}\left[1+(-1)^i\cos\frac{\pi z}{a}\right], z \in \left[0, \frac{na}{2}\right] \cup \left[\frac{(n+2)a}{2}, L\right] \\ 1 + \cos\frac{a\pi}{L} + \frac{1}{2}\left(1-\cos\frac{a\pi}{L}\right)\left(1-\cos\frac{2\pi z}{a}\right), z \in \left[\frac{na}{2}, \frac{(n+2)a}{2}\right] \end{cases} \tag{3b}$$

where i=1, 2,..., n+2 and a is the distance between the adjacent batten plates, $a = L/(n+1)$.

Eq. (3) has clear shortcomings: it is difficult to identify a suitable φ analytically due to the complicated terms. And φ is different for even or odd batten numbers, yielding in complicated computation of the global buckling capacity. For unevenly battened channel columns, φ could be even more difficult to obtain. Considering these limitations, a piecewise cubic-power Hermite interpolation (PCHI) function is proposed here for unevenly battened columns.

For the jth batten interval, $[z_j, z_{j+1}]$ $(0 \le j \le n)$, the PCHI can be expressed as:

$$\Phi_j(z) = g_{1j}(z)\varphi(z_j) + g_{2(j+1)}(z)\varphi(z_{j+1}) + g_{3j}(z)h_j\varphi'(z_j) + g_{4(j+1)}(z)h_j\varphi'(z_{j+1}) \tag{4}$$

where $\Phi_j(z)$ represents the rotating angle between adjacent battens. $\varphi(z_j)$ and $\varphi'(z_j)$ ($1 \le j \le n$) are the values of the original rotation function and its first-order derivative on the mid-line of each batten plate, following the third assumption in Section 2. And $\varphi'(z_0)$ and $\varphi'(z_{n+1})$ will follow the boundary cases, represented by function φ shown in Table 1. With z_0=0, z_{n+1}=L, and h_j=z_{j+1}-z_j, the expressions for g are defined as:

$$g_{1j}(z) = (1+2\frac{z-z_j}{h_j})(1-\frac{z-z_j}{h_j})^2, \quad g_{2(j+1)}(z) = (1-2\frac{z-z_{j+1}}{h_j})(1+\frac{z-z_{j+1}}{h_j})^2$$

$$g_{3j}(z) = \frac{z-z_j}{h_j}(1-\frac{z-z_j}{h_j})^2, \qquad g_{4(j+1)}(z) = \frac{z-z_{j+1}}{h_j}(1+\frac{z-z_{j+1}}{h_j})^2$$

The PCHI, containing the parameters h_j and z_j, allows the continuity in not only φ, but also its first and second-order derivatives for calculating the potential energy of the column.

The global buckling analysis of an unevenly battened column

For an unevenly battened symmetrical column subjected to an eccentrical compressive load P (as shown in Fig. 1), the total potential energy in the form of PCHI is given by:

$$\Pi = \Pi_1+\Pi_2+\Pi_3 \tag{5}$$

where Π_1 is the flexural strain energy and potential energy produced by displacement functions $u(z)$ and $v(z)$,

$$\Pi_1 = \frac{1}{2}\int_0^L \left(EJ_y u''^2 + EJ_x v''^2 - Pu'^2 - Pv'^2 \right) dz \ ;$$

Π_2 is the torsional strain energy and external potential energy produced by the PCHI rotation function $\Phi_j(z)$,

$$\Pi_2 = \frac{1}{2}\sum_{j=0}^{n} \int_{z_j}^{z_{j+1}} \left(EJ_w \Phi_j''^2 + GJ_d \Phi_j'^2 - Pr_e^2 \Phi_j'^2 \right) dz \ ;$$

Π_3 is the torsional potential energy produced by $\Phi_j(z)$, $u(z)$ and $v(z)$,

$$\Pi_3 = \sum_{j=0}^{n} \int_{z_j}^{z_{j+1}} \left[Pe_y u' \Phi_j' + P(x_0 - e_x) v' \Phi_j' \right] dz \ .$$

J_x and J_y are the second moments of area about the x- and y- axes of the channel column without batten plates. J_d and J_w are the torsion and warping constant. A is the cross-sectional area of the column. e_x and e_y are the eccentricity of load P. The single and double apostrophes indicate differentiations with respect to the z-coordinate. Also,

$$r_e^2 = x_0^2 + \frac{J_x + J_y}{A} + \beta_x e_y + \beta_y e_x \,, \quad \beta_x = \frac{1}{J_x} \int_A (y^3 + x^2 y) dA \,, \quad \beta_y = \frac{1}{J_y} \int_A (x^3 + xy^2) dA - 2x_0$$

where r_e is the polar rotating radius of the cross-section about the shear center. β_x and β_y are both asymmetrical coefficients of the cross-section.

For BC Case I, the total potential energy function Π can be calculated as:

$$\begin{aligned} \Pi = & \frac{\pi^2}{4L} C_1^2 \left[EJ_y \left(\frac{\pi}{L} \right)^2 - P \right] + \frac{\pi^2}{4L} C_2^2 \left[EJ_x \left(\frac{\pi}{L} \right)^2 - P \right] + \frac{EJ_w C_3^2}{2} K_1 \\ & + \frac{GJ_d - Pr_e^2}{2} C_3^2 K_2 + \frac{P\pi}{L} \left(e_y C_1 + x_0 C_2 - e_x C_2 \right) C_3 K_3 \end{aligned} \tag{6}$$

Detailed derivation of Eq. (6) is given in Appendix A. Here K_1 is the coefficient of warping torsion; K_2 the coefficient of uniform torsion and K_3 the coefficient of loading. They are given in Appendix A, together with those of the other two BC Cases, respectively.

Applying the Rayleigh-Ritz approach, the non-zero solution of the critical buckling load P can be obtained by setting the determinant (Appendix B) of the governing equations to zero. For simplicity, P is assumed to be applied in the symmetric plane, therefore e_y=0. And the expanded form leads to:

$$(P_y - P)[(P_x - P)(P_x s^2 - Pr_e^2) - \alpha P^2 (x_0 - e_x)^2] = 0 \tag{7}$$

Here P_x and P_y are the pure flexural buckling loads of unbattened columns about x- and y- axis, as shown in Appendix B. α is the buckling coefficient of unevenly battened columns, defined as

$\alpha=2K_3^2/(LK_2)$. Also $s^2=\lambda_x^2 J_w/(Al_w^2)+\lambda_x^2 GJ_d/(A\pi^2 E)$, in which λ_x is the pure flexural buckling slenderness of the column with respect to the x-axis, and l_w is the calculated length of pure torsional buckling of unevenly battened columns, given by $l_w = \pi\sqrt{K_2 / K_1}$

When $r_e \neq \sqrt{\alpha}\left|x_0 - e_x\right|$, the roots of Eq. (7) are:

$$P_1 = P_y \tag{8}$$

$$P_2 = \frac{P_x}{\dfrac{r_e^2+s^2}{2s^2} + \sqrt{\left(\dfrac{r_e^2+s^2}{2s^2}\right)^2 - \dfrac{r_e^2-\alpha\left(x_0-e_x\right)^2}{s^2}}} \tag{9}$$

$$P_3 = \frac{P_x}{\dfrac{r_e^2+s^2}{2s^2} - \sqrt{\left(\dfrac{r_e^2+s^2}{2s^2}\right)^2 - \dfrac{r_e^2-\alpha\left(x_0-e_x\right)^2}{s^2}}} \tag{10}$$

Note that P_2 and P_3 are both torsional-flexural buckling loads. The minimum value of the three solutions is the elastic critical buckling load of the unevenly battened column, and it can be deduced from the comparison between P_1 and the smaller one of P_2 and P_3. To do this, the function P_w and a parabolic function $f(P)$ are applied:

$$P_w = \frac{1}{r_e^2}\left(\frac{\pi^2 EJ_w}{l_w^2} + GJ_d\right) \tag{11}$$

$$f(P) = \left[1-\alpha\left(\frac{x_0-e_x}{r_e}\right)^2\right]P^2 - \left(P_w+P_x\right)P + P_w P_x \tag{12}$$

where P_w is the pure torsional buckling loading of the unevenly battened column, having $r_e^2 P_w = s^2 P_x$.

The following values can be obtained:

$$f(0) = P_w P_x/[1-\alpha(x_0-e_x)^2/r_e^2],$$

$$f(P_w) = -P_w^2[\alpha(x_0-e_x)^2/r_e^2]/[1-\alpha(x_0-e_x)^2/r_e^2],$$

$$f(P_x) = -P_x^2[\alpha(x_0-e_x)^2/r_e^2]/[1-\alpha(x_0-e_x)^2/r_e^2].$$

As mentioned earlier, we need to find the smaller one of P_2 and P_3. Two situations are considered:

(1) $r_e > \sqrt{\alpha}\left|x_0 - e_x\right|$. Here $f(0)$ is positive, both $f(P_w)$ and $f(P_x)$ are negative, and $0<P_2<P_3$.

(2) $r_e < \sqrt{\alpha}\left|x_0 - e_x\right|$. $f(0)$ is negative, both $f(P_w)$ and $f(P_x)$ are positive, and $P_3<0<P_2$. However, $|P_2|<|P_3|$.

Therefore, P_2 is the smaller one and the corresponding slenderness can be obtained as:

$$\lambda = \lambda_x \sqrt{\frac{r_e^2 + s^2}{2s^2} + \sqrt{\left(\frac{r_e^2 + s^2}{2s^2}\right)^2 - \frac{r_e^2 - \alpha\left(x_0 - e_x\right)^2}{s^2}}} \tag{13}$$

When $r_e = \sqrt{\alpha}\left|x_0 - e_x\right|$, the torsional-flexural buckling load P_2 and the corresponding slenderness of unevenly battened columns are:

$$P_2 = P_x \frac{s^2}{r_e^2 + s^2} \tag{14}$$

$$\lambda = \lambda_x \sqrt{1 + \frac{r_e^2}{s^2}} \tag{15}$$

Note that P_2 is always smaller than P_3, and the buckling mode of the unevenly battened column depends only upon P_1 and P_2. If P_2 is smaller than P_1, the column will buckle in the torsional-flexural mode with P_2 being the critical buckling load. Otherwise, buckling will be in the pure flexural mode about y-axis with P_1 being the critical flexural buckling load.

In similar approaches, derivations of the buckling coefficient α, the calculated length l_w, the critical buckling load P and the corresponding slenderness λ for BC Cases II and III can be obtained, together with variable r_j used for the calculation of K_i.

It is noted that in the PCHI approach parameters K_i (i=1, 2, 3) do not need different formulae for even or odd number of batten plates. This is more practical and effective than the classic analytical approach (Wang 1985).

It is noted that for unevenly battened columns, the buckling coefficient α is always positive under BC Cases I and III, therefore the denominator of Eq. (9) is always bigger than 1, which means the torsional-flexural buckling critical load P_2 is always smaller than the pure flexural buckling critical load P_x. Hence, for BC Cases I and III, the buckling will always be in the torsional-flexural mode only if $P_x<P_y$.

In the following section, the PCHI solutions of both evenly and unevenly battened columns are verified numerically with the classic analytical approach (Wang 1985) and finite element modeling, respectively.

Numerical verification of the proposed PCHI method

Verification of evenly battened columns with the classic analytical approach

The buckling coefficient α and the torsional-flexural slenderness ratio obtained from PCHI method were compared with those from the classic analytical (CA) approach (Wang 1985) under the three BC Cases (Table 1) considered for evenly battened columns, in which a single-symmetric channel column made of steel (Young elastic modulus E = 210 GPa and Poisson's ratio ν = 0.3) was analyzed. The column has width b = 20cm, height h = 15cm, flange thickness t_1 = 2cm, web thickness t_2 = 3cm and the moments of inertia J_x = 5415.4cm^4 and J_y = 5637.9cm^4, respectively. The axial compressive load P has an eccentricity e_x = 10cm and e_y = 0cm.

The buckling coefficients α obtained from the PCHI and CA methods are compared in terms of the battens' number in Table 2. The results show that the proposed PCHI results are good agreement with those of the CA approach. Furthermore, it may be worth noting that the obtained results for evenly battened columns show better agreement with CA's results when the number of battens is odd in BC Cases I and III.

Fig.2 illustrates the normalized torsional-flexural slenderness ratios λ/λ_x vs. the number of batten plates for the three BC Cases considered. The results show that results of the PCHI method agrees well with those of the CA (Wang 1985) approach in λ of evenly battened columns, confirming the validity of the proposed PCHI method. Fig. 2 also shows that with the increase of the battens' number, buckling tends to become a pure flexural mode rather than a mixed torsional-flexural one due to the increased torsional stiffness produced by battens.

***Verification of unevenly battened column with Finite Element Analysis* (*FEA*)**

To verify the efficiency of the proposed PCHI method in calculating the global buckling capacity of unevenly battened columns, columns braced by 2, 3 and 4 battens were modeled, respectively, under three aforementioned BC Cases. Positions of battens were chosen in three groups at locations of intervals of one-tenth of the total column length shown in Fig. 3. Nine dots equally divided along the column axis indicate the possible positions of battens. We considered the following scenarios in FEA: the shaded battens are fixed at their positions, and one batten (shown as hollow) is placed at various free dot positions. This allows us to check the influence of both the batten numbers and positions.

The battened column was modeled as made of steel, having a span L=100cm, a width b=14cm, a height h=10cm, the flanges and web thickness t_1=t_2=2cm and the width of battens b_p=5cm and the thickness t_p=0.25cm. 4-node quadrilateral shell element (Shell63) provided by ANSYS was applied. A convergence test showed that 650 to 674 elements were appropriate, as shown in Fig.4, depending on the number of battens included in the battened model. To avoid local warping at the column ends, two end cross-sections were defined rigid with a "master" node at the centroid (node 1 in Fig.4 insert) linked rigidly to a number of "slave" nodes on the flanges and web. The column model was axially loaded by a compressive force *P* at node 1 (shown in Fig.4). A linear buckling analysis was carried out. The calculated first buckling modes of with 2, 3 and 4 battens positioned along the column axis are illustrated in Fig.5. The numerical modes were verified with the experimental work of Sahoo and Rai (2007) and Gosowski (2007).

Fig.6 shows the torsional-flexural buckling forces under different BC cases from the proposed PCHI method and FEA, with the horizontal *z*-axis indicating the position of the "free" batten. Good agreement between two results is evident. The buckling load of BC Case III (fully clamped) is much higher than those of BC Cases I and II (both hinged). And the buckling load of BC Case II is the lowest for no warping constraints at the ends. It is also interesting to see that only BC Case III is very sensitive to batten positions. It shows large variations in the buckling force with the lowest value occurring when the "free" batten is placed around the mid of the column span. However, with the increase of the battens' number, the sensitivity shows a decreasing trend.

Influence of load eccentricities

A parameter study was performed in order to examine the effect of the eccentricities e_x and e_y on the torsional-flexural buckling capacity of evenly and unevenly battened columns, respectively. The effect of e_x on the buckling load is considered first, followed by the mixed effects of both e_x and e_y. The battened column model used for parametric study was the same as in section 5.2.

The effect of e_x

Using the proposed PCHI approach, the normalized torsional-flexural buckling loads of evenly and unevenly battened columns were calculated in terms of the eccentricity e_x for BC Case I and are given in Figs. 7 and 8, respectively. Fig. 7 shows the influence of the load eccentricity in *evenly battened columns* with different number of batten plates, indicated by n. Taking the centroid position as the origin as in Fig. 1, the axial load was positioned from the negative e_x to the positive one. It can be seen in Fig. 7 that with the reducing eccentricity in the negative domain, the critical buckling load goes up in magnitude in a combined torsional-flexural mode with decreasing torsional deformation. The critical buckling load reaches its highest value in the pure flexural mode at $e_x/|x_0|= -1$. It then goes down when e_x is further reduced to zero and then increased in the positive domain, and the buckling mode returns back in the combined torsional-flexural one with increasing torsional deformation.

The influence of batten number n is also illustrated in Fig. 7. With increased n, the critical buckling load becomes higher due to the increased stiffness by battens, except at the pure flexural mode where the critical buckling load remains unchanged. The change of the critical buckling load also becomes gradual in terms of $e_x/|x_0|$.

Fig.8 shows the influence of e_x on *unevenly battened column* of two battens under BC Case I, with one batten fixed at position $L/5$ (or $2L/10$) and the other "free" batten placed at different positions as indicated by z. While the buckling load shows a similar up and down trend in terms of e_x as in the evenly battened case, a key feature is the higher buckling load when the free batten is placed towards the column ends, and lower towards the middle. This is in good agreement with Fig.6a.

For the case of the compressive load applied with an eccentricity, the relation between the torsional-flexural buckling load P_{cr} and the corresponding moment M_{cr} ($M_{cr} = P_{cr}\,|x_0 - e_x|$) can be deduced from Eq. (7) and notice that the eccentricity $|x_0 - e_x|$ is also variable with the buckling load P_{cr}.

$$\left(1-\frac{P_{cr}}{P_x}\right)\left(1-\frac{P_{cr}}{P_w}\right)-\alpha\frac{{M_{cr}}^2}{{M_{cry}}^2}=0 \tag{16}$$

Here $M_{cry}=\sqrt{r_e^{\,2}P_xP_w}$, being the pure flexural buckling moment of unevenly battened columns. The relationship between P_{cr} and M_{cr} in Eq. (16) is illustrated in Figs. 9 and 10 in terms of the buckling ratios P_w/P_x, which represents the comparison between the torsional and flexural rigidities. Figs. 9a and 10a show the decrement of torsional-flexural buckling load from the Euler buckling resistance (M = 0) to the pure flexural buckling ($M = M_{cr}$) for *evenly* battened columns and *unevenly* battened columns of two battens under BC Case I, respectively. With the effect of buckling coefficient α, the column remains displaying the torsional-flexural buckling mode at terminal point ($M = M_{cr}$). Notice that at $M = 0$, $P_{cr}/P_x = 1$ for $P_w/P_x = 1$ and 5 and $P_{cr}/P_x = 0.2$ for $P_w/P_x = 0.2$.

For evenly battened columns in Fig.9, columns reinforced with more battens have larger moment M_{cr} and eccentricity $|x_0 - e_x|$ under the same value of the buckling load P_{cr}. The effects of batten's number on the eccentricity $|x_0 - e_x|$ appears less significant as shown in the inserts of Fig.9b. For unevenly battened columns in Fig.10, columns with the 'free' batten placed towards the column ends show larger moment M_{cr} and eccentricity $|x_0 - e_x|$ to buckle than those with the 'free' batten towards the middle when subjected to the same buckling load P_{cr}.

Figs. 9 and 10 can be used for design application, for instance, to choose the eccentricity e_x of the loading, and calculate the corresponding lateral moment M_{cr} to satisfy the required torsional-flexural buckling load P_{cr} for evenly and unevenly battened columns, respectively.

The effect of combined eccentricity e_x and e_y

Applying the Eq. (32) in Appendix B, the combined effect of loading eccentricities e_x and e_y on the torsional-flexural buckling load is shown in Figs. 11 and 12 for evenly and unevenly battened columns, respectively.

Fig. 11a shows the surface of the buckling load P_{cr} of evenly battened columns reinforced by 2, 5 and 9 battens under BC Case I, respectively. The torsional-flexural buckling load increases in terms of the battens' number except at the apex, where the buckling mode changes into the pure flexural one with the load applied at the shear center of the column's section. Fig. 11b shows the sectional view in the plane e_y=0. The profile curves are exactly those shown in Fig. 7. For the section view at the plane $e_x/|x_0| = -1$ in Fig. 11c, all buckling surfaces display symmetrical characteristic with the plane $e_y = 0$ since the column's section is singly-symmetrical with the x-axis. Fig. 11d shows contours of the buckling load P_{cr} in e_x-e_y plane. For each batten number, the curvature of the contour loops increases in terms of P_{cr}, and the loop center moves towards positive e_x. For the same P_{cr}/P_x value, contour loops become larger with increased batten's number, suggesting that with more battens, a larger eccentricity is needed to buckle if the axial load remains the same. In other words, additional battens will enhance the buckling capacity of the column. Similarly, one may draw the similar conclusion that with increased battens, the critical buckling load will be larger if the load eccentricity remains the same.

Fig.12a shows the surface of the buckling load P_{cr} of unevenly battened columns under BC Case I reinforced by two battens with the 'fixed' batten located at $2L/10$ and the 'free' batten at $L/10$, $6L/10$ and $9L/10$, respectively. The torsional-flexural buckling load is higher when the "free" batten is placed at the end of the column except at the apex, where buckling becomes a pure flexural mode. The sectional curves through the plane e_y=0 in Fig.12b are exactly the ones shown in Fig.8. And for the section view at the plane $e_x/|x_0| = -1$ shown in Fig.12c, all buckling surfaces display symmetrical characteristics, similar to Fig.11c. Fig.12d shows the contours of the buckling load P_{cr} in e_x-e_y plane. The curvature and center of the contours show the same character as those in Fig.11d. Also, it is observed that with the 'free' batten moves from the middle to the end of column, the contour becomes larger, consistent with Figs.6a and 8. Analogical suggestions can be obtained: a column braced by two battens located towards the column's ends needs a larger eccentricity of the axial load to buckle. One may draw the similar conclusion that with the 'free' batten placed towards the column ends, the critical buckling load will be larger if the load eccentricity remains the same.

Considering the combined effect of eccentricities e_x *and* e_y, the relation of the torsional-flexural buckling load P_{cr} and the corresponding moments M_x ($P_{cr}|e_y|$) and M_y ($P_{cr}|x_0 - e_x|$) can also be obtained in terms of the buckling ratio P_w/P_x and the distance ratio $\eta = |x_0 - e_x|/|e_y|$, representing the relative extent of eccentricity. Eq. (17) is deduced from Eq. (32) in Appendix B in which both the load eccentricities e_y and $|x_0 - e_x|$ are variables of the buckling load P_{cr}:

$$\left(1-\frac{P_{cr}}{P_y}\right)\left(1-\frac{P_{cr}}{P_x}\right)\left(1-\frac{P_{cr}}{P_w}\right)=\alpha\left(1-\frac{P_{cr}}{P_x}\right)\frac{M_x^{\ 2}}{M_{crx}^{\ 2}}+\alpha\left(1-\frac{P_{cr}}{P_x}\frac{J_x}{J_y}\right)\frac{M_y^{\ 2}}{M_{cry}^{\ 2}} \tag{17}$$

Here $M_{crx}=\sqrt{r_e^{\,2}P_yP_w}$ and $M_{cry}=\sqrt{r_e^{\,2}P_xP_w}$, are the pure flexural buckling moments about x- and y-axis of a unevenly battened column. The relationship between the buckling load P_{cr} and the corresponding moment M_x of evenly and unevenly battened columns are depicted in Figs.13 and 14 with $J_x/J_y = 1$, respectively. Four parameters are considered below – the buckling ratio P_w/P_x aforementioned in 6.1, the distance ratio η, the number and displacement of batten plates.

M_x of evenly battened columns is illustrated in Fig. 13(a), (b) and (c), as grouped by the distance ratios η (0.2, 1 and 5) for three P_w/P_x values. The intersection between the curves and the y-axis shows a positive correlation with the batten plate number (in all Fig. 13(a), (b) and (c)) and a negative correlation with the distance ratio η (across Fig. 13 (a), (b) and (c)). When loaded in compression, M_x $/M_{crx}$ decreases and becomes zero at the Euler buckling with $P_{cr}/P_x = 1$ for $P_w/P_x = 1$ and 5, and 0.2 for $P_w/P_x = 0.2$. Under tension, M_x increases in terms of P until the column reaches its tensile strength. Also, it can be seen that columns with a higher buckling ratio have higher M_x under a compressive load, but a lower M_x under a tensile load.

M_x of unevenly battened columns is illustrated in Fig.14. As in the evenly battened case, one may draw similar conclusions in terms of the “free” batten plate position.

Conclusions

Considering the potential to enhance the global buckling capacity of battened columns by changing the spacing between battens, the channel columns braced by unevenly distributed batten plates were investigated under three boundary conditions (Table 1). Solutions for the torsional-flexural buckling critical loads of unevenly battened columns were obtained using the proposed PCHI method. The present analysis highlights the effect of unevenly distributed battens on the global torsional-flexural buckling capacity of battened columns. Based on the analysis, main conclusions can be drawn as listed below:

(1) Elastic torsional-flexural buckling loads can be obtained using the proposed PCHI method for unevenly battened columns subjected to axially compressive loading with eccentricities. The coefficients of warping torsion K_1, uniform torsion K_2 and the coefficient of loading K_3 remain unchanged, irrespective to even or odd number of the batten plates used, for the three different boundary conditions discussed, respectively. This is clearly advantageous when compared to the

classical analytical approach where different formulae have to be used for odd and even number of battens.

(2) The results of the new PCHI method proposed here show good agreement with the classic analytical solutions for evenly battened columns and with the finite element analysis for unevenly battened columns in the torsional-flexural buckling load under the three BC Cases discussed. With the battens' number increased, the buckling mode of the column tends to become a pure flexural mode rather than a mixed torsional-flexural one. Furthermore, the buckling load of BC Case III (fully clamped) is very sensitive to batten positions. It shows large variations in the buckling force with the lowest value occurring when the "free" batten is placed around the mid of the column span. However, with the increase of the battens' number, this sensitivity diminishes.

(3) In the parameter study, the effects of double-eccentricities e_x and e_y on the torsional-flexural buckling load of both evenly and unevenly battened columns were described in a 3D surface. The maximum buckling load (the apex of the 3D surface) occurs when the compressive load is applied at the shear center of the column's cross-section, and the buckling mode changed into a pure flexural mode rather than a mixed torsional-flexural buckling one. With a reducing buckling load, the curvatures of the ellipse-shaped contour loops reduce and the center of the contour moves further into the negative eccentricity e_x. For the even case, when the batten number is increased, the curvature of the buckling load contour also reduces. This means that a column braced by more battens needs a larger loading eccentricity to buckle when the external load remains the same. For the uneven case, with the 'free' batten moves from the middle to the end of column, the contour becomes larger. This means that a column braced by two battens located towards the column's ends needs a larger eccentricity of the axial load to buckle when the external load remains the same. Furthermore,

relationships between the buckling load P_{cr} and the corresponding moments M_x and M_y for various buckling ratios (P_w/P_x) and distance ratios η were obtained. The influential effects of some factors on the relationship, such as the batten number and their positions, were discussed.

Based on PCHI approach, it is possible to improve the global buckling capacity of battened columns through optimizing the spacing amongst battens. The future work will be extensively researched for (1) the continuity of the inner bi-moments and shear force of the battened column; (2) the effects of shear stiffness of battens on torsional and flexural stiffness of columns; (3) flange local buckling, distortional buckling and their interactions.

Appendix A Derivation of the total potential energy *Π*As listed in Table 1, the displacement and rotation functions in BC Case I are expressed as:

$$u=C_1\sin(\pi z/L) \tag{18}$$

$$v=C_2\sin(\pi z/L) \tag{19}$$

$$\varphi(z_j)=C_3r_j,(1\leq j \leq n). \tag{20}$$

in which r_j=1-cos($2\pi z_j/L$).

Substitute Eqs. (18)~ (19) into Π_1 of Eq. (5):

$$\Pi_1=\frac{1}{2}\int_0^L\left(EJ_yu''^2+EJ_xv''^2-Pu'^2-Pv'^2\right)dz=\frac{\pi^2}{4L}C_1^2\left[EJ_y\left(\frac{\pi}{L}\right)^2-P\right]+\frac{\pi^2}{4L}C_2^2\left[EJ_x\left(\frac{\pi}{L}\right)^2-P\right] \tag{21}$$

Note that the third and fourth assumption, substitute the PCHI function (Eq. (4)) and Eqs. (18)~ (20) into Π_2 and Π_3 of Eq. (5):

$$\begin{aligned}\Pi_2 &= \frac{1}{2}\sum_{j=0}^{n}\int_{z_j}^{z_{j+1}}\left(EJ_w\Phi_j''^2 + GJ_d\Phi_j'^2 - Pr_e^2\Phi_j'^2\right)dz \\ &= \frac{EJ_w}{2}\sum_{j=0}^{n}\int_{z_j}^{z_{j+1}}\left[\varphi(z_j)g_{1j}''(z)+\varphi(z_{j+1})g_{2(j+1)}''(z)\right]^2 dz + \frac{GJ_d - Pr_e^2}{2}\sum_{j=0}^{n}\int_{z_j}^{z_{j+1}}\left[\varphi(z_j)g_{1j}'(z)+\varphi(z_{j+1})g_{2(j+1)}'(z)\right]^2 dz \\ &= C_3^2\frac{EJ_w}{2}\sum_{j=0}^{n}\int_{z_j}^{z_{j+1}}\left[r_j g_{1j}''(z)+r_{j+1}g_{2(j+1)}''(z)\right]^2 dz + C_3^2\frac{GJ_d - Pr_e^2}{2}\sum_{j=0}^{n}\int_{z_j}^{z_{j+1}}\left[r_j g_{1j}'(z)+r_{j+1}g_{2(j+1)}'(z)\right]^2 dz\end{aligned} \quad (22)$$

And,

$$\begin{aligned}\Pi_3 &= \sum_{j=0}^{n}\int_{z_j}^{z_{j+1}}\left[Pe_y u'\Phi_j' + P(x_0 - e_x)v'\Phi_j'\right]dz \\ &= \frac{\pi P\left[e_y C_1 + (x_0 - e_x)C_2\right]}{L}\sum_{j=0}^{n}\int_{z_j}^{z_{j+1}}\Phi_j'\cos\frac{\pi z}{L}dz \\ &= \frac{\pi PC_3\left[e_y C_1 + (x_0 - e_x)C_2\right]}{L}\sum_{j=0}^{n}\int_{z_j}^{z_{j+1}}\left[g_{1j}'(z)r_j + g_{2(j+1)}'(z)r_{j+1}\right]\cos\frac{\pi z}{L}dz\end{aligned} \quad (23)$$

Note that $g_{2(j+1)}'(z)= -g_{1j}'(z)$ and $g_{2(j+1)}''(z)= -g_{1j}''(z)$, the Eqs. (22) and (23) can be translated into:

$$\Pi_2 = C_3^2\frac{EJ_w}{2}\sum_{j=0}^{n}\Delta_j^2\int_{z_j}^{z_{j+1}}g_{1j}''^2(z)dz + C_3^2\frac{GJ_d - Pr_e^2}{2}\sum_{j=0}^{n}\Delta_j^2\int_{z_j}^{z_{j+1}}g_{1j}'^2(z)dz \quad (24)$$

And,

$$\Pi_3 = -\frac{\pi PC_3\left[e_y C_1 + (x_0 - e_x)C_2\right]}{L}\sum_{j=0}^{n}\Delta_j\int_{z_j}^{z_{j+1}}g_{1j}'(z)\cos\frac{\pi z}{L}dz \quad (25)$$

where Δ_j being the first-order forward difference of r_j.

Combine Eqs. (21), (24) and (25) into total potential energy (Eq. (5)), we have:

$$\begin{aligned}\Pi = {} & \frac{\pi^2}{4L}C_1^2\left[EJ_y\left(\frac{\pi}{L}\right)^2 - P\right] + \frac{\pi^2}{4L}C_2^2\left[EJ_x\left(\frac{\pi}{L}\right)^2 - P\right] + \frac{EJ_w C_3^2}{2}\sum_{j=0}^{n}\Delta_j^2\int_{z_j}^{z_{j+1}}g_{1j}''^2 dz \\ & + \frac{\left(GJ_d - Pr_e^2\right)C_3^2}{2}\sum_{j=0}^{n}\Delta_j^2\int_{z_j}^{z_{j+1}}g_{1j}'^2 dz - \frac{\pi PC_3\left[e_y C_1 + (x_0 - e_x)C_2\right]}{L}\sum_{j=0}^{n}\Delta_j\int_{z_j}^{z_{j+1}}g_{1j}'\cos\frac{\pi z}{L}dz\end{aligned} \quad (26)$$

So the derivation of Eq. (6) is obtained in detail from Eqs. (18)~ (26) and by comparing Eq. (6) with (26), coefficients K_i (i=1,2,3) can be obtained with $K_1 = \sum_{j=0}^{n}\Delta_j^2\int_{z_j}^{z_{j+1}}g_{1j}''^2 dz$, $K_2 = \sum_{j=0}^{n}\Delta_j^2\int_{z_j}^{z_{j+1}}g_{1j}'^2 dz$,

$K_3 = -\sum_{j=0}^{n} \mathit{\Delta}_j \int_{z_j}^{z_{j+1}} \cos\frac{\pi z}{L} g'_{1j} dz$. Additionally, the analogy procedure can be performed for BC Cases II and III.

For BC Case II:

$$K_1 = \sum_{j=0}^{n} \int_{z_j}^{z_{j+1}} \left[\left(g''_{1j} r_j + g''_{4(j+1)} h_j r'_{j+1} \right)^2 + \left(g''_{2(j+1)} r_{j+1} + g''_{3j} h_j r'_j \right)^2 + 2 g''_{1j} r_j g''_{2(j+1)} r_{j+1} \right] dz \tag{27a}$$

$$K_2 = \sum_{j=0}^{n} \int_{z_j}^{z_{j+1}} \left[\left(g'_{1j} r_j + g'_{4(j+1)} h_j r'_{j+1} \right)^2 + \left(g'_{2(j+1)} r_{j+1} + g'_{3j} h_j r'_j \right)^2 + 2 g'_{1j} r_j g'_{2(j+1)} r_{j+1} \right] dz \tag{27b}$$

$$K_3 = \sum_{j=0}^{n} \int_{z_j}^{z_{j+1}} \cos\frac{\pi z}{L} \left(g'_{1j} r_j + g'_{2(j+1)} r_{j+1} + g'_{3j} h_j r'_j + g'_{4(j+1)} h_j r'_{j+1} \right) dz \tag{27c}$$

in which $r_j = \sin(\pi z_j / L)$

And for BC Case III:

$$K_1 = \sum_{j=0}^{n} \mathit{\Delta}_j^2 \int_{z_j}^{z_{j+1}} g''^{\,2}_{1j} dz \tag{28a}$$

$$K_2 = \sum_{j=0}^{n} \mathit{\Delta}_j^2 \int_{z_j}^{z_{j+1}} g'^{\,2}_{1j} dz \tag{28b}$$

$$K_3 = -\sum_{j=0}^{n} \mathit{\Delta}_j \int_{z_j}^{z_{j+1}} \sin\frac{2\pi z}{L} g'_{1j}(z) dz \tag{28c}$$

where $\mathit{\Delta}_j$ is the first-order forward difference of r_j.and $r_j = 1 - \cos(2\pi z_j / L)$.

Appendix B Non-zero solution *P* obtained from Rayleigh-Ritz approach

Applying the Rayleigh-Ritz approach on Eq. (6), the variations of Eq. (6) with respect to C_i (i=1,2,3) results in the following algebraic equations:

$$C_1 \frac{\pi^2}{2L} \left[EJ_y \left(\frac{\pi}{L} \right)^2 - P \right] + C_3 \frac{P\pi}{L} e_y K_3 = 0 \tag{29}$$

$$C_2 \frac{\pi^2}{2L} \left[EJ_x \left(\frac{\pi}{L} \right)^2 - P \right] + C_3 \frac{P\pi}{L} \left(x_0 - e_x \right) K_3 = 0 \tag{30}$$

$$\frac{P\pi}{L}K_3e_yC_1+\frac{P\pi}{L}K_3\left(x_0-e_x\right)C_2+C_3\left[EJ_wK_1+\left(GJ_d-Pr_e^2\right)K_2\right]=0 \tag{31}$$

The non-zero solution of the critical buckling load P can be obtained by making the determinant into zero, and the expanded form leads to:

$$\left(P_y-P\right)\left(P_x-P\right)\left(P_xs^2-Pr_e^2\right)-\alpha P^2e_y^2\left(P_x-P\right)-\alpha P^2\left(P_y-P\right)\left(x_0-e_x\right)^2=0 \tag{32}$$

where $P_x=EJ_x(\pi/L)^2$, $P_y=EJ_y(\pi/L)^2$ for BC Cases I and II; $P_x=EJ_x(2\pi/L)^2$, $P_y=EJ_y(2\pi/L)^2$ for BC Cases III.

So when the external load P is applied in the symmetrical plane, e_y=0, and the Eq. (7) can be obtained.

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Notation

The following symbols are used in this paper:

A = cross-sectional area of the column;
a = distance between adjacent battens in evenly battened columns;
b = width of top and bottom flanges;
b_p = width of batten plates;
C_1, C_2, C_3 = coefficient constants to be determined;
E = young elastic modulus;
e_x, e_y = eccentricities along the x- and y-axis, respectively;
G = shear modulus;
h = Height of the web;
J_d = torsional inertia;
J_w = warping constant;
J_x, J_y = moments of inertia about x- and y-axis, respectively;
K_1 = coefficient of warping torsion;
K_2 = coefficient of uniform torsion;

K_3 = coefficient of loading;
L = span of the column;
l_w = length of pure torsional buckling, $l_w=\pi\sqrt{K_2/K_1}$;
n = number of batten plates;
P = axial force;
P_x, P_y = euler critical loads about the x- and y-axis, respectively;
P_w = pure torsional buckling loading produced by axial compressive load through shear center;
r_j = rotation function with respect to z_j;
r_e = rolar rotating radius of the cross-section;
t_1 = thickness of the flanges;
t_2 = thickness of the web;
t_p = thickness of batten plates;
u, v = displacements along x- and y-axis with respect to the shear center;
x_0 = x-coordinate of shear center;
z = position of the "free" batten;
z_j = positions of batten plates;
α = buckling coefficient of unevenly battened columns;
λ_x = flexural buckling slenderness about x-axis;
φ = rotation function of shear center;
Π = total potential energy of system;
Δ_j = first-order forward difference of r_j;
$\Phi_j(z)$ = piecewise hermite interpolation function in $[z_j, z_{j+1}]$;

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Fig. 1. (a) the measurements of cross-sectional profile (b) the battened column (c) deformations of cross section after buckling. The original and buckling positions of cross section A-A are shown in dashed and solid lines, respectively

Fig. 2. The solid line represents the PCHI method, and dot-dash line represents CA method; circle, diamond and fork represent the span of 50cm, 200cm and 500cm, respectively

Fig. 3. The blank rectangular represents the 'free' batten and hatched one represents 'fixed' battens; the black circles is the locations the free batten may take

Fig. 4. The insert shows the rigid boundary condition at both ends (BC Case III) using the "master-slave node technique" (master at the centroid)

Fig. 5. "Exaggerated" first buckling meshing modes with (a) 2, (b) 3 and (c) 4 battens positioned along the column axis

Fig. 6. The dashed line represents FEA method and the solid line represents PCHI method; fork, diamond and circle represent boundary conditions I, II and III, respectively

Fig. 7. Eight types of lines represent eight batten's number in evenly batten columns, where x_0 is the x-coordinate of the shear centre and n is the number of battens

Fig. 8. Eight types of lines represent eight possible positions of the 'free' battens in unevenly batten columns, where x_0 is the x-coordinate of the shear centre and z is the position of the 'free' batten as indicated in Fig. 3

Fig. 9. (a) four types of lines represent four batten's number, and all lines are grouped in terms of the buckling ratios P_w/P_x (b) three types of lines represent three batten's number

Fig. 10. (a) five types of lines represent five positions of 'free' batten, and all lines are grouped in terms of the buckling ratios P_w/P_x (b) three types of lines represent three positions of 'free' batten

Fig. 11. (a) the dashed, thick solid and solid lines represent the case of evenly battened columns with 2, 5 and 9 battens, respectively (b, c, d) represent the corresponding section view at the plane $e_y/|x_0| = 0$ and $e_x/|x_0| = -1$ and the contour curves. Numbers shown are the values of P_{cr}/P_x

Fig. 12. (a) the dashed, thick solid and solid lines represent the case of unevenly battened columns with the 'free' batten located at '6L/10', '9L/10' and 'L/10', respectively (b, c, d) represent the corresponding section view at the plane $e_y/|x_0| = 0$ and $e_x/|x_0| = -1$ and the contour curves. Numbers shown are the values of P_{cr}/P_x

Fig. 13. Dashed and solid lines represent the evenly battened columns with 2 and 10 battens, respectively; all curves are grouped in terms of the buckling ratios P_w/P_x

Fig. 14. Dashed, dot and solid lines represent the unevenly battened columns with the 'free' batten located at 'L/10', '6L/10' and '9L/10', respectively

Table 1. Cases of Boundary Conditions and Their Corresponding Displacement Functions u, v *and* φ.

BC Cases	BC		function u	function v	function φ
I. HR	$u=v=\varphi=0$	$u''=v''=\varphi'=0$	$C_1 \sin\frac{\pi z}{L}$	$C_2 \sin\frac{\pi z}{L}$	$C_3\left(1-\cos\frac{2\pi z}{L}\right)$
II. HF	$u=v=\varphi=0$	$u''=v''=\varphi''=0$	$C_1 \sin\frac{\pi z}{L}$	$C_2 \sin\frac{\pi z}{L}$	$C_3 \sin\frac{\pi z}{L}$
III. FR	$u=v=\varphi=0$	$u'=v'=\varphi'=0$	$C_1\left(1-\cos\frac{2\pi z}{L}\right)$	$C_2\left(1-\cos\frac{2\pi z}{L}\right)$	$C_3\left(1-\cos\frac{2\pi z}{L}\right)$

Note: BC = Boundary condition; HR = Hinged supports with warping deformations restrained; HF = Hinged supports with warping deformations free; FR = Fixed supports with warping deformations restrained.

Table 2. The Buckling Coefficient α of Three BC Cases

BC Cases	Method	The number of batten plates											
		2	3	4	5	6	7	8	9	10	...	50	100
I	PCHI	0.789	0.690	0.654	0.636	0.626	0.620	0.616	0.613	0.610	...	0.601	0.601
	CA	0.646	0.672	0.614	0.619	0.602	0.603	0.595	0.596	0.592	...	0.584	0.584
	ε (%)	18.12	2.61	6.12	2.67	3.83	2.74	3.41	2.77	2.95	...	2.83	2.83
II	PCHI	0.679	0.723	0.748	0.764	0.775	0.783	0.789	0.794	0.798	...	0.826	0.830
	CA	0.805	0.787	0.803	0.8	0.806	0.805	0.807	0.807	0.808	...	0.810	0.811
	ε (%)	18.56	8.85	7.35	4.71	4.00	2.81	2.28	1.64	1.25	...	1.94	2.29
III	PCHI	0.667	0.736	0.770	0.789	0.8	0.808	0.813	0.817	0.820	...	0.833	0.833
	CA	0.614	0.721	0.733	0.769	0.772	0.787	0.788	0.796	0.796	...	0.810	0.810
	ε (%)	8.63	2.08	5.05	2.60	3.63	2.67	3.17	2.64	3.02	...	2.84	2.84

Note: PCHI = the piecewise cubic Hermite interpolation; CA = the classic analytical approach (Wang 1985); ε = the relative error between PCHI method and CA method.